

THE DUBOVITSKIĬ-SARD THEOREM IN SOBOLEV SPACES

PIOTR HAJLASZ AND SCOTT ZIMMERMAN

ABSTRACT. The Sard theorem from 1942 requires that a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^k , $k > \max(n - m, 0)$. In 1957 Duvovitskiĭ generalized Sard's theorem to the case of C^k mappings for all k . Namely he proved that, for almost all $y \in \mathbb{R}^m$, $\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0$ where $\ell = \max(n - m - k + 1, 0)$, \mathcal{H}^ℓ denotes the Hausdorff measure, and C_f is the set of critical points of f . In 2001 De Pascale proved that the Sard theorem holds true for Sobolev mappings of the class $W_{\text{loc}}^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$, $k > \max(n - m, 0)$ and $p > n$. We will show that also Dubovitskiĭ's theorem can be generalized to the case of $W_{\text{loc}}^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ mappings for all $k \in \mathbb{N}$ and $p > n$.

1. INTRODUCTION

Originally proven in 1942, Arthur Sard's [23] famous theorem asserts that the set of critical values of a sufficiently regular mapping is null. We will use the following notation to represent the *critical set* of a given smooth map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$C_f = \{x \in \mathbb{R}^n \mid \text{rank } Df(x) < m\}.$$

Throughout this paper, we will assume that m and n are integers at least 1. Most notation used in the introduction is carefully explained in Section 2.

Theorem 1.1 (Sard). *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^k . If $k > \max(n - m, 0)$, then*

$$\mathcal{H}^m(f(C_f)) = 0.$$

Here and in what follows by \mathcal{H}^k we denote the k -dimensional Hausdorff measure.

Several results have shown that Sard's result is optimal, see e.g. [7, 12, 13, 15, 19, 27]. In 1957 Dubovitskiĭ [7], extended Sard's theorem to all orders of smoothness k . See [3] for a modernized proof of this result and some generalizations.

Theorem 1.2 (Dubovitskiĭ). *Fix $n, m, k \in \mathbb{N}$. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^k . Write $\ell = \max(n - m - k + 1, 0)$. Then*

$$\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.$$

This result tells us that almost every level set of a smooth mapping intersects with its critical set on an ℓ -null set. Higher regularity of the function implies a reduction in the Hausdorff dimension of the overlap between $f^{-1}(y)$ and C_f for a.e. $y \in \mathbb{R}^m$.

2010 *Mathematics Subject Classification.* 46E35, 58C25.

Key words and phrases. Sard theorem, Sobolev spaces, Kneser-Glaeser theorem.

P.H. was supported by NSF grant DMS-1161425.

Notice that if $k > \max(n - m, 0)$, then $n - m - k + 1 \leq 0$, and so $\mathcal{H}^\ell = \mathcal{H}^0$ is simply the counting measure on \mathbb{R}^n . That is, if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^k and additionally $k > \max(n - m, 0)$, Dubovitskiĭ's theorem implies that $f^{-1}(y) \cap C_f$ is empty for almost every $y \in \mathbb{R}^m$. In other words, $\mathcal{H}^m(f(C_f)) = 0$. Thus Sard's theorem is a special case of Dubovitskiĭ's theorem.

Recently, many mathematicians have worked to generalize Sard's result to the class of Sobolev mappings [2, 3, 4, 5, 6, 9, 17, 26]. Specifically, in 2001 De Pascale [6] proved the following version of Sard's theorem for Sobolev mappings.

Theorem 1.3. *Suppose $k > \max(n - m, 0)$. Suppose $\Omega \subset \mathbb{R}^n$ is open. If $f \in W_{\text{loc}}^{k,p}(\Omega, \mathbb{R}^m)$ for $n < p < \infty$, then $\mathcal{H}^m(f(C_f)) = 0$.*

In this paper we will use the usual notation $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ to indicate the Sobolev class of $L^p(\mathbb{R}^n, \mathbb{R}^m)$ mappings whose first k weak partial derivatives have finite L^p norm.

The purpose of this paper is to show that also the Dubovitskiĭ theorem generalizes to the case of $W_{\text{loc}}^{k,p}$ mappings when $n < p < \infty$. We must be very careful when dealing with Sobolev mappings because the set $f^{-1}(y)$ depends on what representative of f we take. If $k \geq 2$, then Morrey's inequality implies that f has a representative of class $C^{k-1, 1-\frac{n}{p}}$, so the critical set C_f is well defined. If $k = 1$, then Df is only defined almost everywhere and hence the set C_f is defined up to a set of measure zero. We will say that f is *precisely represented* if each component f_i of f satisfies

$$f_i(x) = \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f_i(y) dy$$

for all $x \in \Omega$ at which this limit exists. The Lebesgue differentiation theorem ensures that this is indeed a well defined representative of f . In what follows, we will always refer to the $C^{k-1, 1-\frac{n}{p}}$ representative of f when $k \geq 2$ and a precise representation of f when $k = 1$. (Notice that the precise representative of f and the smooth representative of f are the same for $k \geq 2$.)

The main result of the paper reads as follows.

Theorem 1.4. *Fix $n, m, k \in \mathbb{N}$. Suppose $\Omega \subset \mathbb{R}^n$ is open and $f \in W_{\text{loc}}^{k,p}(\Omega, \mathbb{R}^m)$ for some $n < p < \infty$. If $\ell = \max(n - m - k + 1, 0)$, then*

$$\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.$$

If $m > n$, then since $p > n$ we may apply Morrey's inequality combined with Hölder's inequality to show that $\mathcal{H}^n(f(Q)) < \infty$ for any cube $Q \Subset \Omega$, and so $\mathcal{H}^m(f(\Omega)) = 0$. Thus $f^{-1}(y)$ is empty for almost every $y \in \mathbb{R}^m$, and the theorem follows.

We will now discuss the details behind the argument that $\mathcal{H}^n(f(Q)) < \infty$ for any cube $Q \Subset \Omega$. Fix $\delta > 0$, and cover Q with $2^{n\nu}$ congruent dyadic cubes $\{Q_j\}_{j=1}^{2^{n\nu}}$ with pairwise disjoint interiors. According to Morrey's inequality (see Lemma 2.3),

$$\text{diam } f(Q_j) \leq C(\text{diam } Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} |Df(z)|^p dz \right)^{1/p}$$

for every $1 \leq j \leq 2^{n\nu}$. Since $\text{diam } Q_j = 2^{-\nu} \text{diam } Q$, choosing ν large enough gives $\sup_j \text{diam } f(Q_j) < \delta$, and so we can estimate the pre-Hausdorff measure

$$\begin{aligned} \mathcal{H}_\delta^n(f(Q)) &\leq C \sum_{j=1}^{2^{n\nu}} (\text{diam } f(Q_j))^n \\ &\leq C \sum_{j=1}^{2^{n\nu}} (\text{diam } Q_j)^{n(1-\frac{n}{p})} \left(\int_{Q_j} |Df(z)|^p dz \right)^{n/p} \\ &\leq C \left(\sum_{j=1}^{2^{n\nu}} (\text{diam } Q_j)^n \right)^{1-\frac{n}{p}} \left(\sum_{j=1}^{2^{n\nu}} \int_{Q_j} |Df(z)|^p dz \right)^{n/p} \\ &\leq C \mathcal{H}^n(Q)^{1-\frac{n}{p}} \left(\int_Q |Df(z)|^p dz \right)^{n/p}. \end{aligned}$$

We used Hölder's inequality with exponents p/n and $p/(p-n)$ to obtain the third line. Since the right hand estimate does not depend on δ , sending $\delta \rightarrow 0^+$ yields $\mathcal{H}^n(f(Q)) < \infty$. This completes the proof of Theorem 1.4 when $m > n$. Hence we may assume that $m \leq n$.

We will now discuss the case $k = 1$ to avoid any confusion involving the definition of C_f . Since $m \leq n$, we may apply the following co-area formula due to Malý, Swanson, and Ziemer [19]:

Theorem 1.5. *Suppose that $1 \leq m \leq n$, $\Omega \subset \mathbb{R}^n$ is open, $p > m$, and $f \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^m)$ is precisely represented. Then the following holds for all measurable $E \subset \Omega$:*

$$\int_E |J_m f(x)| dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(E \cap f^{-1}(y)) dy$$

where $|J_m f|$ is the square root of the sum of the squares of the determinants of the $m \times m$ minors of Df .

Notice that $|J_m f|$ is equal to zero almost everywhere on the set $E = C_f$. Therefore the above equality with $E = C_f$ reads

$$0 = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(C_f \cap f^{-1}(y)) dy = \int_{\mathbb{R}^m} \mathcal{H}^\ell(C_f \cap f^{-1}(y)) dy.$$

That is, $\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$, and the theorem follows.

Therefore, we may assume for the remainder of the paper that $m \leq n$ and $k \geq 2$.

Most proofs of Sard-type results typically involve some form of a Morse Theorem [22] in which the critical set of a mapping is decomposed into pieces on which the function's difference quotients converge quickly. See [24] for the proof of the classical Sard theorem based on this method. A version of the Morse Theorem was also used by De Pascale [6]. However, there is another approach to the Sard theorem based on the so called Kneser-Glaeser Rough Composition theorem, and this method entirely avoids the use of the Morse theorem. We say that a mapping $f : W \subset \mathbb{R}^r \rightarrow \mathbb{R}^m$ of class C^k is s -flat on $A \subset W$ for $1 \leq s \leq k$ if $D^\alpha f = 0$ on A for every $1 \leq |\alpha| \leq s$.

Theorem 1.6 (Kneser-Glaeser Rough Composition). *Fix positive integers s, k, r, n with $s < k$. Suppose $V \subset \mathbb{R}^r$ and $W \subset \mathbb{R}^n$ are open. Let $g : V \rightarrow W$ be of class C^{k-s} and $f : W \rightarrow \mathbb{R}$ be of class C^k . Suppose $A^* \subset V$ and $A \subset W$ are compact sets with*

- (1) $g(A^*) \subset A$ and
- (2) f is s -flat on A .

Then there is a function $F : \mathbb{R}^r \rightarrow \mathbb{R}$ of class C^k so that $F = f \circ g$ on A^ and F is s -flat on A^* .*

This theorem ensures that the composition of two smooth maps will have the same regularity as the second function involved in the composition provided that enough of the derivatives of this second function are zero. After a brief examination of the rule for differentiation of composite functions, such a conclusion seems very natural. Indeed, we can formally compute $D^\alpha(f \circ g)(x)$ for all $|\alpha| \leq k$ and $x \in A^*$ since any “non-existing” derivative $D^\beta g(x)$ with $|\beta| > k - s$ is multiplied by a vanishing $D^\gamma f(g(x))$ term with $|\gamma| = |\alpha| - |\beta| < s$. Thus we can formally set $D^\gamma f(g(x))D^\beta g(x) = 0$. However the proof of this theorem is not easy since it is based on the celebrated Whitney extension theorem. That should not be surprising after all. The existence of the extension F is proven by verification that the formal jet of derivatives of $f \circ g$ up to order k defined above satisfies the assumptions of the Whitney extension theorem.

In 1951, Kneser presented a proof of this composition result in [16]. In the same paper, he proved a theorem which may be obtained as an immediate corollary to the theorem of Sard, though he did so without any reference to or influence from Sard’s result. The composition theorem is also discussed in a different context in a 1958 paper by Glaeser [11]. The reader may find the proofs of this theorem in [1, Theorem 14.1], [18, Chapter 1, Theorem 6.1], [20, Theorem 8.3.1].

Thom [25], quickly realized that the method of Kneser can be used to prove the Sard theorem. See also [1, 18, 21]. Recently Figalli [9] used this method to provide a simpler proof of Theorem 1.3. Our proof of Theorem 1.4 we will also be based on the Kneser-Glaeser result.

2. NOTATION AND AUXILIARY RESULTS

In this section we will explain notation and prove some technical results related to the Morrey inequality that will be used in the proof of Theorem 1.4.

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$. By $D^\alpha f$ we will denote the partial derivative of f with respect to the multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. In particular $D^{\delta_i} f = \partial f / \partial x_i$, i.e. $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a multiindex with 1 on i th position. Also $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. $D^k f$ will denote the vector whose components are the derivatives $D^\alpha f$, $|\alpha| = k$. The classes of functions with continuous and α -Hölder continuous derivatives of order up to k will be denoted by C^k and $C^{k,\alpha}$ respectively. The integral average over a set S of positive measure will be denoted by

$$f_S = \fint_S f(x) dx = \frac{1}{|S|} \int_S f(x) dx.$$

The characteristic function of a set E will be denoted by χ_E . The k -dimensional Hausdorff measure will be denoted by \mathcal{H}^k . In particular \mathcal{H}^0 is the counting measure. The Lebesgue measure in \mathbb{R}^n coincides with \mathcal{H}^n . In addition to the Hausdorff measure notation we will also write $|S|$ for the Lebesgue measure of S . We say that a set is k -null if its k -dimensional Hausdorff measure equals zero. By \mathcal{H}_δ^k , $\delta > 0$, we denote the pre-Hausdorff measure defined by taking infimum over coverings of the set by sets of diameters less than δ so $\mathcal{H}^k(E) = \lim_{\delta \rightarrow 0+} \mathcal{H}_\delta^k(E)$. Cubes in \mathbb{R}^n will always have sides parallel to coordinate directions. The symbol C will be used to represent a generic constant and the actual value of C may change in a single string of estimates. By writing $C = C(n, m)$ we indicate that the constant C depends on n and m only.

We will use the following elementary result several times.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be a bounded measurable set and let $-\infty < a < n$. Then there is a constant $C = C(n, a)$ such that for every $x \in E$*

$$\int_E \frac{dy}{|x - y|^a} \leq \begin{cases} C|E|^{1-\frac{a}{n}} & \text{if } 0 \leq a < n. \\ (\text{diam } E)^{-a}|E| & \text{if } a < 0. \end{cases}$$

Proof. The case $a < 0$ is obvious since then $|x - y|^{-a} \leq (\text{diam } E)^{-a}$. Thus assume that $0 \leq a < n$. In this case the inequality is actually true for all $x \in \mathbb{R}^n$ and not only for $x \in E$. Let $B = B(0, r)$, $|B| = |E|$. We have

$$\int_E \frac{dy}{|x - y|^a} \leq \int_B \frac{dy}{|y|^a} = C \int_0^r t^{-a} t^{n-1} dt = Cr^{n-a} = C|E|^{1-\frac{a}{n}}.$$

□

The following result [10, Lemma 7.16] is a basic pointwise estimate for Sobolev functions.

Lemma 2.2. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{1,p}(D)$, $p \geq 1$, then*

$$(2.1) \quad |f(x) - f_S| \leq C(n) \frac{|D|}{|S|} \int_D \frac{|Df(z)|}{|x - z|^{n-1}} dz \quad \text{a.e.}$$

When $p > n$, the triangle inequality $|f(y) - f(x)| \leq |f(y) - f_D| + |f(x) - f_D|$, Hölder inequality, and Lemma 2.1 applied to the right hand side of (2.1) yield a well known

Lemma 2.3 (Morrey's inequality). *Suppose $n < p < \infty$ and $f \in W^{1,p}(D)$, where $D \subset \mathbb{R}^n$ is a cube or a ball. Then there is a constant $C = C(n, p)$ such that*

$$|f(y) - f(x)| \leq C(\text{diam } D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p dz \right)^{1/p} \quad \text{for all } x, y \in D.$$

In particular,

$$\text{diam } f(D) \leq C(\text{diam } D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p dz \right)^{1/p}.$$

Since $p > n$, the function f is continuous (Sobolev embedding) and hence the lemma does indeed hold for *all* $x, y \in D$.

From this lemma we can easily deduce a corresponding result for higher order derivatives. The Taylor polynomial and the averaged Taylor polynomial of f will be denoted by

$$T_x^k f(y) = \sum_{|\alpha| \leq k} D^\alpha f(x) \frac{(y-x)^\alpha}{\alpha!}, \quad T_S^k f(y) = \int_S T_x^k f(y) dx.$$

Lemma 2.4. *Suppose $n < p < \infty$, $k \geq 1$ and $f \in W^{k,p}(D)$, where $D \subset \mathbb{R}^n$ is a cube or a ball. Then there is a constant $C = C(n, k, p)$ such that*

$$|f(y) - T_x^{k-1} f(y)| \leq C(\text{diam } D)^{k-\frac{n}{p}} \left(\int_D |D^k f(z)|^p dz \right)^{1/p} \quad \text{for all } x, y \in D.$$

Proof. Given $y \in D$ let

$$\psi(x) := T_x^{k-1} f(y) = \sum_{|\alpha| \leq k-1} D^\alpha f(x) \frac{(y-x)^\alpha}{\alpha!} \in W^{1,p}(D).$$

Observe that $\psi(y) = f(y)$ and

$$\frac{\partial \psi}{\partial x_j}(x) = \sum_{|\alpha| = k-1} D^{\alpha + \delta_j} f(x) \frac{(y-x)^\alpha}{\alpha!},$$

where $\delta_j = (0, \dots, 1, \dots, 0)$. Indeed, after applying the Leibniz rule to $\partial \psi / \partial x_j$ the lower order terms will cancel out. Since

$$|D\psi(z)| \leq C(n, k) |D^k f(z)| |y - z|^{k-1},$$

Lemma 2.3 applied to ψ yields the result. \square

Applying the same argument to Lemma 2.2 leads to the following result, see [3, Theorem 3.3].

Lemma 2.5. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{k,p}(D)$, $p \geq 1$, $k \geq 1$, then there is constant $C = C(n, k)$ such that*

$$(2.2) \quad |f(x) - T_S^{k-1} f(x)| \leq C \frac{|D|}{|S|} \int_D \frac{|D^k f(z)|}{|x - z|^{n-k}} dz \quad \text{for a.e. } x \in D.$$

In the next result we will improve the above estimates under the additional assumption that the derivative Df vanishes on a given subset of D . For a similar result in a different setting see [14, Proposition 2.3].

Lemma 2.6. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $f \in W^{k,p}(D)$, $n < p < \infty$, $k \geq 1$. Let*

$$A = \{x \in D \mid Df(x) = 0\}.$$

Then for any $\varepsilon > 0$ there is $\delta = \delta(n, k, p, \varepsilon) > 0$ such that if

$$\frac{|D \setminus A|}{|D|} < \delta,$$

then

$$\text{diam } f(D) \leq \varepsilon (\text{diam } D)^{k-\frac{n}{p}} \left(\int_D |D^k f(z)|^p dz \right)^{1/p}.$$

Remark 2.7. It is important that δ does not depend of f . The result applies very well to density points of A . Indeed, it follows immediately that if $x \in A$ is a density point, then for any $\varepsilon > 0$ there is $r_x > 0$ such that

$$\text{diam } f(B(x, r_x)) \leq \varepsilon r_x^{k-\frac{n}{p}} \left(\int_{B(x, r_x)} |D^k f(z)|^p dz \right)^{1/p}.$$

Proof of Lemma 2.6. Although only the first order derivatives of f are equal zero in A , it easily follows that $D^\alpha f = 0$ a.e. in A for all $1 \leq |\alpha| \leq k$. Indeed, if a Sobolev function is constant in a set, its derivative equals zero a.e. in the set, [10, Lemma 7.7], and we apply induction. Hence

$$T_A^{k-1} f(x) = f_A \quad \text{for all } x \in \mathbb{R}^n.$$

Let $\varepsilon > 0$. Choose $0 < \delta < 1/2$ with $\max \left\{ \delta^{\frac{k}{n}-\frac{1}{p}}, \delta^{1-\frac{1}{p}} \right\} < \varepsilon$. Since $\delta < 1/2$, $|D|/|A| < 2$. Thus Lemma 2.5 with $S = A$ yields

$$|f(x) - f_A| \leq C(n) \int_{D \setminus A} \frac{|D^k f(z)|}{|x - z|^{n-k}} dz \leq C(n) \|D^k f\|_{L^p(D)} \left(\int_{D \setminus A} \frac{dz}{|x - z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}}.$$

Now the result follows directly from Lemma 2.1. Indeed, if $k \leq n$, Lemma 2.1 and the estimate

$$|D \setminus A| < \delta |D| \leq C(n) \delta (\text{diam } D)^n$$

yield

$$\left(\int_{D \setminus A} \frac{dz}{|x - z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \leq C(n, k, p) |D \setminus A|^{\frac{1}{n}(k-\frac{n}{p})} \leq C(n, k, p) \delta^{\frac{k}{n}-\frac{1}{p}} (\text{diam } D)^{k-\frac{n}{p}}.$$

If $k > n$, then we have

$$\left(\int_{D \setminus A} \frac{dz}{|x - z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \leq (\text{diam } D)^{k-n} |D \setminus A|^{\frac{p-1}{p}} \leq C(n, p) \delta^{1-\frac{1}{p}} (\text{diam } D)^{k-\frac{n}{p}}.$$

Hence

$$\text{diam } f(D) = \sup_{x, y \in D} |f(x) - f(y)| \leq 2 \sup_{x \in D} |f(x) - f_A| \leq C(n, k, p) \varepsilon (\text{diam } D)^{k-\frac{n}{p}} \|D^k f\|_{L^p(D)}.$$

The proof is complete. □

We will also need the following classical Besicovitch covering lemma, see e.g. [28, Theorem 1.3.5]

Lemma 2.8 (Besicovitch). *Let $E \subset \mathbb{R}^n$ and let $\{B_x\}_{x \in E}$ be a family of closed balls $B_x = \overline{B}(x, r_x)$ so that $\sup_{x \in E} \{r_x\} < \infty$. Then there is a countable (possibly finite) subfamily $\{B_{x_i}\}_{i=1}^\infty$ with the property that*

$$E \subset \bigcup_{i=1}^{\infty} B_{x_i}$$

and no point of \mathbb{R}^n belongs to more than $C(n)$ balls.

3. PROOF OF THEOREM 1.4

As we pointed out in Introduction we may assume that $m \leq n$ and $k \geq 2$. It is also easy to see that we can assume that $\Omega = \mathbb{R}^n$ and $f \in W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$. Indeed, it suffices to prove the claim of Theorem 1.4 on compact subsets of Ω and so we may multiply f by a compactly supported smooth cut-off function to get a function in $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$.

We will prove the result using induction with respect to n .

If $n = 1$, then $m = n = 1$. This gives $n - m - k + 1 = 1 - k \leq 0$ for any $k \in \mathbb{N}$, so $\ell = 0$. Thus the theorem is a direct consequence of Theorem 1.5.

We shall prove now the theorem for $n \geq 2$ assuming that it is true in dimensions less than or equal to $n - 1$.

Fix p and integers m and k satisfying $n < p < \infty$, $m \leq n$, and $k \geq 2$. Write $\ell = \max(n - m - k + 1, 0)$. Let $f \in W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$.

We can write

$$C_f = K \cup A_1 \cup \dots \cup A_{k-1},$$

where

$$K := \{x \in C_f \mid 0 < \text{rank } Df(x) < m\}$$

and

$$A_s := \{x \in \mathbb{R}^n \mid D^\alpha f(x) = 0 \text{ for all } 1 \leq |\alpha| \leq s\}$$

Note that $A_1 \supset A_2 \supset \dots \supset A_{k-1}$ is a decreasing sequence of sets.

In the first step, we will show that $A_{k-1} \cap f^{-1}(y)$ is ℓ -null for a.e. $y \in \mathbb{R}^m$. Then we will prove the same for $(A_{s-1} \setminus A_s) \cap f^{-1}(y)$ for $s = 2, 3, \dots, k-1$. To do this we will use the Implicit Function and Kneser-Glaeser theorems to reduce our problem to a lower dimensional one and apply the induction hypothesis. Finally, we will consider the set K and use a change of variables to show that we can reduce the dimension in the domain and in the target so that the fact that $\mathcal{H}^\ell(K \cap f^{-1}(y)) = 0$ will follow from the induction hypothesis.

Claim 3.1. $\mathcal{H}^\ell(A_{k-1} \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Proof. Suppose $x \in A_{k-1}$. Notice that $T_x^{k-1}f(y) = f(x)$ for any $y \in \mathbb{R}^n$ since $D^\alpha f(x) = 0$ for every $1 \leq |\alpha| \leq k-1$. By Lemma 2.4 applied to each coordinate of $f = (f_1, \dots, f_m)$,

we have for any cube $Q \subset \mathbb{R}^n$ containing x and any $y \in Q$,

$$(3.1) \quad |f(y) - f(x)| \leq C(\text{diam } Q)^{k-\frac{n}{p}} \left(\int_Q |D^k f(z)|^p dz \right)^{1/p}.$$

Hence

$$(3.2) \quad \text{diam } f(Q) \leq C(\text{diam } Q)^{k-\frac{n}{p}} \left(\int_Q |D^k f(z)|^p dz \right)^{1/p}.$$

Let $F_1 := \{x \in A_{k-1} \mid x \text{ is a density point of } A_{k-1}\}$ and $F_2 := A_{k-1} \setminus F_1$. We will treat the sets $F_1 \cap f^{-1}(y)$ and $F_2 \cap f^{-1}(y)$ separately.

Step 1. First we will prove that $\mathcal{H}^\ell(F_2 \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.

Let $0 < \varepsilon < 1$. Since $\mathcal{H}^n(F_2) = 0$, there is an open set $F_2 \subset U \subset \Omega$ such that $\mathcal{H}^n(U) < \varepsilon^{\frac{p}{p-m}}$. For any $j \geq 1$ let $\{Q_{ij}\}_{i=1}^\infty$ be a collection of closed cubes with pairwise disjoint interiors such that

$$Q_{ij} \cap F_2 \neq \emptyset, \quad F_2 \subset \bigcup_{i=1}^\infty Q_{ij} \subset U, \quad \text{diam } Q_{ij} < \frac{1}{j}.$$

Since $F_2 \cap Q_{ij} \neq \emptyset$, (3.2) yields

$$\mathcal{H}^m(f(Q_{ij})) \leq C(\text{diam } f(Q_{ij}))^m \leq C(\text{diam } Q_{ij})^{m(k-\frac{n}{p})} \left(\int_{Q_{ij}} |D^k f(x)|^p dx \right)^{m/p}.$$

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

This condition easily implies that $mk \geq n$ so we also have $\frac{mp}{p-m}(k - \frac{n}{p}) \geq n$, and by Hölder's inequality,

$$\begin{aligned} \mathcal{H}^m(f(F_2)) &\leq \sum_{i=1}^\infty \mathcal{H}^m(f(Q_{ij})) \leq C \sum_{i=1}^\infty (\text{diam } Q_{ij})^{m(k-\frac{n}{p})} \left(\int_{Q_{ij}} |D^k f(x)|^p dx \right)^{m/p} \\ &\leq C \left(\sum_{i=1}^\infty (\text{diam } Q_{ij})^{\frac{pm}{p-m}(k-\frac{n}{p})} \right)^{\frac{p-m}{p}} \left(\int_{\bigcup_{i=1}^\infty Q_{ij}} |D^k f(x)|^p dx \right)^{m/p} \\ (3.3) \quad &\leq C \mathcal{H}^n(U)^{\frac{p-m}{p}} \left(\int_U |D^k f(x)|^p dx \right)^{m/p} < C\varepsilon \|D^k f\|_p. \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, $\mathcal{H}^m(f(F_2)) = 0$ and hence $F_2 \cap f^{-1}(y) = \emptyset$, i.e. $\mathcal{H}^\ell(F_2 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\{Q_{ij} \cap f^{-1}(y)\}_{i=1}^\infty$ form a covering of $F_2 \cap f^{-1}(y)$ by sets of diameters less than $1/j$. Since

$$\text{diam}(Q_{ij} \cap f^{-1}(y)) \leq (\text{diam } Q_{ij}) \chi_{f(Q_{ij})}(y)$$

the definition of the Hausdorff measure yields

$$\begin{aligned}
(3.4) \quad \mathcal{H}^\ell(F_2 \cap f^{-1}(y)) &\leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^{\infty} \text{diam}(Q_{ij} \cap f^{-1}(y))^\ell \\
&\leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^{\infty} (\text{diam } Q_{ij})^\ell \chi_{f(Q_{ij})}(y).
\end{aligned}$$

We would like to integrate both sides with respect to $y \in \mathbb{R}^m$. Note that the function on the right hand side is measurable since the sets $f(Q_{ij})$ are compact. However measurability of the function $y \mapsto \mathcal{H}^\ell(F_2 \cap f^{-1}(y))$ is far from being obvious. To deal with this problem we will use the *upper integral* which for a non-negative function $g : X \rightarrow [0, \infty]$ defined μ -a.e. on a measure space (X, μ) is defined as follows:

$$\int_X^* g \, d\mu = \inf \left\{ \int_X \phi \, d\mu : 0 \leq g \leq \phi \text{ and } \phi \text{ is } \mu\text{-measurable} \right\}.$$

An important property of the upper integral is that if $\int_X^* g \, d\mu = 0$, then $g = 0$ μ -a.e. Indeed, there is a sequence $\phi_i \geq g \geq 0$ such that $\int_X \phi_i \, d\mu \rightarrow 0$. That means $\phi_i \rightarrow 0$ in $L^1(\mu)$. Taking a subsequence we get $\phi_{i_j} \rightarrow 0$ μ -a.e. which proves that $g = 0$ μ -a.e.

Applying the upper integral with respect to $y \in \mathbb{R}^m$ to both sides of (3.4), using Fatou's lemma, and noticing that

$$\frac{p}{p-m} \left(\ell + m \left(k - \frac{n}{p} \right) \right) \geq n$$

gives

$$\begin{aligned}
\int_{\mathbb{R}^m}^* \mathcal{H}^\ell(F_2 \cap f^{-1}(y)) \, d\mathcal{H}^m(y) &\leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^{\infty} (\text{diam } Q_{ij})^\ell \mathcal{H}^m(f(Q_{ij})) \\
&\leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^{\infty} (\text{diam } Q_{ij})^{\ell+m(k-\frac{n}{p})} \left(\int_{Q_{ij}} |D^k f(x)|^p \, dx \right)^{m/p} < C\varepsilon \|D^k f\|_p^m
\end{aligned}$$

by the same argument as in (3.3). Again, since $\varepsilon > 0$ can be arbitrarily small, we conclude that $\mathcal{H}^\ell(F_2 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Step 2. It remains to prove that $\mathcal{H}^\ell(F_1 \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.

The proof is similar to that in Step 1 and the arguments which are almost the same will be presented in a more sketchy form now. In Step 1 it was essential that the set F_2 had measure zero. We will compensate the lack of this property now by the estimates from Remark 2.7.

It suffices to prove that for any cube \tilde{Q} , $\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. Assume that \tilde{Q} is in the interior of a larger cube $\tilde{Q} \Subset Q$.

By Remark 2.7, for each $x \in \tilde{Q} \cap F_1$ and $j \in \mathbb{N}$ there is $0 < r_{jx} < 1/j$ such that

$$\text{diam } f(B(x, r_{jx})) \leq j^{-1} r_{jx}^{k-\frac{n}{p}} \left(\int_{B(x, r_{jx})} |D^k f(z)|^p \, dz \right)^{1/p}.$$

We may further assume that $B(x, r_{jx}) \subset Q$.

Denote $B_{jx} = \overline{B}(x, r_{jx})$. According to the Besicovitch Lemma 2.8, there is a countable subcovering $\{B_{jx_i}\}_{i=1}^\infty$ of $\tilde{Q} \cap F_1$ so that no point of \mathbb{R}^n belongs to more than $C(n)$ balls B_{jx_i} .

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

We have $\frac{pm}{p-m}(k - \frac{n}{p}) \geq n$ as before, so

$$\begin{aligned} \mathcal{H}^m(f(\tilde{Q} \cap F_1)) &\leq C \sum_{i=1}^\infty \mathcal{H}^m(f(B_{jx_i})) \leq C j^{-m} \sum_{i=1}^\infty r_{jx_i}^{m(k - \frac{n}{p})} \left(\int_{B_{jx_i}} |D^k f(z)|^p dz \right)^{m/p} \\ &\leq C j^{-m} \left(\sum_{i=1}^\infty r_{jx_i}^n \right)^{\frac{p-m}{p}} \left(\sum_{i=1}^\infty \int_{B_{jx_i}} |D^k f(z)|^p dz \right)^{m/p}. \end{aligned}$$

Since the balls are contained in Q and no point belongs to more than $C(n)$ balls we conclude that

$$\mathcal{H}^m(f(\tilde{Q} \cap F_1)) \leq C j^{-m} \mathcal{H}^n(Q)^{\frac{p-m}{p}} \|D^k f\|_p^m.$$

Since j can be arbitrarily large, $\mathcal{H}^m(f(\tilde{Q} \cap F_1)) = 0$, i.e. $\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\{B_{jx_i} \cap f^{-1}(y)\}_{i=1}^\infty$ form a covering of $\tilde{Q} \cap F_1 \cap f^{-1}(y)$ and

$$\text{diam}(B_{jx_i} \cap f^{-1}(y)) \leq C r_{jx_i} \chi_{f(B_{jx_i})}(y).$$

The definition of the Hausdorff measure yields

$$\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) \leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^\infty r_{jx_i}^\ell \chi_{f(B_{jx_i})}(y).$$

Thus as above

$$\begin{aligned} &\int_{\mathbb{R}^m}^* \mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) d\mathcal{H}^m(y) \\ &\leq C \liminf_{j \rightarrow \infty} \sum_{i=1}^\infty r_{jx_i}^\ell \mathcal{H}^m(f(B_{jx_i})) \\ &\leq C \liminf_{j \rightarrow \infty} j^{-m} \sum_{i=1}^\infty r_{jx_i}^{\ell + m(k - \frac{n}{p})} \left(\int_{B_{jx_i}} |D^k f(z)|^p dz \right)^{m/p} \\ &\leq C \liminf_{j \rightarrow \infty} j^{-m} \mathcal{H}^n(Q)^{\frac{p-m}{p}} \|D^k f\|_p^m = 0 \end{aligned}$$

since $\frac{p}{p-m} \left(\ell + m \left(k - \frac{n}{p} \right) \right) \geq n$. Therefore $\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

This completes the proof that $\mathcal{H}^\ell(F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$ and hence that of Claim 3.1 \square

Claim 3.2. $\mathcal{H}^\ell((A_{s-1} \setminus A_s) \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$, $s = 2, 3, \dots, k-1$.

In this step, we will use the Kneser-Glaeser composition theorem and the implicit function theorem to apply the induction hypothesis in \mathbb{R}^{n-1} .

Fix $s \in \{2, 3, \dots, k-1\}$ and $\bar{x} \in A_{s-1} \setminus A_s$. It suffices to show that the ℓ -Hausdorff measure of $W \cap (A_{s-1} \setminus A_s) \cap f^{-1}(y)$ is zero for some neighborhood W of \bar{x} and a.e. $y \in \mathbb{R}^m$. Indeed, $A_{s-1} \setminus A_s$ can be covered by countably many such neighborhoods.

By the definitions of A_s and A_{s-1} , $D^\gamma f(\bar{x}) = 0$ for all $1 \leq |\gamma| \leq s-1$, and $D^\beta f(\bar{x}) \neq 0$ for some $|\beta| = s$. That is, for some $|\gamma| = s-1$ and $j \in \{1, \dots, m\}$, $D(D^\gamma f_j)(\bar{x}) \neq 0$ and $D^\gamma f_j \in W^{k-(s-1), p} \subset C^{k-s, 1-\frac{n}{p}}$.

Hence, by the implicit function theorem, there is some neighborhood U of \bar{x} and an open set $V \subset \mathbb{R}^{n-1}$ so that $U \cap \{D^\gamma f_j = 0\} = g(V)$ for some $g : V \rightarrow \mathbb{R}^n$ of class C^{k-s} . In particular, $U \cap A_{s-1} \subset g(V)$ since $D^\gamma f_j = 0$ on A_{s-1} .

Choose a neighborhood $W \Subset U$ of \bar{x} and say $A^* := g^{-1}(\overline{W} \cap A_{s-1})$ so that A^* is compact. Since f is $s-1$ flat on the closed set A_{s-1} , f is of class C^{k-1} , g is of class $C^{(k-1)-(s-1)}$, and $g(A^*) \subset A_{s-1}$, we can apply Theorem 1.6 to each component of f to find a C^{k-1} function $F : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^m$ so that, for every $x \in A^*$, $F(x) = (f \circ g)(x)$ and $D^\lambda F(x) = 0$ for all $|\lambda| \leq s-1$. That is, $A^* \subset C_F$. Hence

$$\mathcal{H}^\ell(A^* \cap F^{-1}(y)) \leq \mathcal{H}^\ell(C_F \cap F^{-1}(y)) = 0.$$

for almost every $y \in \mathbb{R}^m$. In this last equality, we invoked the induction hypothesis on $F \in C^{k-1}(\mathbb{R}^{n-1}, \mathbb{R}^m) \subset W_{\text{loc}}^{k-1, p}(\mathbb{R}^{n-1}, \mathbb{R}^m)$ with $\ell = \max((n-1) - m - (k-1) + 1, 0)$. Since g is of class C^1 , it is locally Lipschitz, and so $\mathcal{H}^\ell(g(A^* \cap F^{-1}(y))) = 0$ for almost every $y \in \mathbb{R}^m$. Since $W \cap A_{s-1} \subset g(A^*)$, we have

$$W \cap A_{s-1} \cap f^{-1}(y) \subset g(A^* \cap F^{-1}(y))$$

for all $y \in \mathbb{R}^m$, and thus

$$\mathcal{H}^\ell(W \cap (A_{s-1} \setminus A_s) \cap f^{-1}(y)) \leq \mathcal{H}^\ell(W \cap A_{s-1} \cap f^{-1}(y)) = 0$$

for almost every $y \in \mathbb{R}^m$. The proof of the claim is complete.

Claim 3.3. $\mathcal{H}^\ell(K \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Proof. Write $K = \bigcup_{r=1}^{m-1} K_r$ where $K_r := \{x \in \mathbb{R}^n \mid \text{rank } Df(x) = r\}$. Fix $x_0 \in K_r$ for some $r \in \{1, \dots, m-1\}$. For the same reason as in Claim 3.2 it suffices to show that $\mathcal{H}^\ell((V \cap K_r) \cap f^{-1}(y)) = 0$ for some neighborhood V of x_0 for a.e. $y \in \mathbb{R}^m$.

Without loss of generality, assume that the submatrix $[\partial f_i / \partial x_j(x_0)]_{i,j=1}^r$ formed by the first r rows and columns of Df has rank r . Let

$$(3.5) \quad Y(x) = (f_1(x), f_2(x), \dots, f_r(x), x_{r+1}, \dots, x_n) \quad \text{for all } x \in \mathbb{R}^n.$$

Y is of class C^{k-1} since each component of f is. Also, $\text{rank } DY(x_0) = n$, so by the inverse function theorem Y is a C^{k-1} diffeomorphism of some neighborhood V of x_0 onto an open set $\tilde{V} \subset \mathbb{R}^n$. From now on we will assume that Y is defined in V only.

Claim 3.4. $Y^{-1} \in W_{\text{loc}}^{k, p}(\tilde{V}, \mathbb{R}^n)$.

Proof. In the proof we will need

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be open. If $g, h \in W_{\text{loc}}^{\ell,p}(\Omega)$, where $p > n$ and $\ell \geq 1$, then $gh \in W_{\text{loc}}^{\ell,p}(\Omega)$.*

Proof. Since $g, h \in C^{\ell-1}$, it suffices to show that the classical partial derivatives $D^\beta(gh)$, $|\beta| = \ell - 1$ belong to $W_{\text{loc}}^{1,p}(\Omega)$ (when $\ell = 1$, $\beta = 0$ so $D^\beta(gh) = gh$).

The product rule for $C^{\ell-1}$ functions yields

$$(3.6) \quad D^\beta(gh) = \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma! \delta!} D^\gamma g D^\delta h.$$

Each of the functions $D^\gamma g, D^\delta h$ is absolutely continuous on almost all lines parallel to coordinate axes, [8, Section 4.9.2], so is their product. Thus $D^\beta(gh)$ is absolutely continuous on almost all lines and hence it has partial derivatives (of order 1) almost everywhere. According to a characterization of $W_{\text{loc}}^{1,p}$ by absolute continuity on lines, [8, Section 4.9.2], it suffices to show that partial derivatives of $D^\beta(gh)$ (of order 1) belong to L_{loc}^p . This will imply that $D^\beta(gh) \in W_{\text{loc}}^{1,p}$ for all β , $|\beta| = \ell - 1$ so $gh \in W_{\text{loc}}^{\ell,p}$.

If $D^\alpha = D^{\delta_i} D^\beta$, then the product rule applied to the right hand side of (3.6) yields

$$D^\alpha(gh) = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma! \delta!} D^\gamma g D^\delta h.$$

If $|\gamma| < |\alpha| = \ell$ and $|\delta| < |\alpha| = \ell$, then the function $D^\gamma g D^\delta h$ is continuous and hence in L_{loc}^p . The remaining terms are $h D^\alpha g + g D^\alpha h$. Clearly this function also belongs to L_{loc}^p because the functions g, h are continuous and $D^\alpha g, D^\alpha h \in L_{\text{loc}}^p$. This completes the proof of the lemma. \square

Now we can complete the proof of Claim 3.4. Since Y is a diffeomorphism of class C^{k-1} , we have

$$(3.7) \quad D(Y^{-1})(y) = [DY(Y^{-1}(y))]^{-1} \quad \text{for every } y \in \tilde{V}.$$

It suffices to prove that $D(Y^{-1}) \in W_{\text{loc}}^{k-1,p}$. It follows from (3.7) and a formula for the inverse matrix that

$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_2(Df)} \right) \circ Y^{-1},$$

where P_1 and P_2 are polynomials whose variables are replaced by partial derivatives of f . The polynomial $P_2(Df)$ is just $\det DY$.

Since $Df \in W_{\text{loc}}^{k-1,p}$ and $p > n$, it follows from Lemma 3.5 that

$$P_1(Df), P_2(Df) \in W_{\text{loc}}^{k-1,p}.$$

Note that $P_2(Df) = \det DY$ is continuous and different than zero. Hence

$$\frac{1}{P_2(Df)} \in W_{\text{loc}}^{k-1,p}$$

as a composition of a $W_{\text{loc}}^{k-1,p}$ function which is locally bounded away from 0 and ∞ with a smooth function $x \mapsto x^{-1}$. Thus Lemma 3.5 applied one more time yields that

$P_1(Df)/P_2(Df) \in W_{\text{loc}}^{k-1,p}$. Finally

$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_2(Df)} \right) \circ Y^{-1} \in W_{\text{loc}}^{k-1,p}$$

because composition with a diffeomorphism Y^{-1} of class C^{k-1} preserves $W_{\text{loc}}^{k-1,p}$. The proof of the claim is complete. \square

It follows directly from (3.5) that

$$(3.8) \quad f(Y^{-1}(x)) = (x_1, \dots, x_r, g(x))$$

for all $x \in \tilde{V}$ and some function $g : \tilde{V} \rightarrow \mathbb{R}^{m-r}$.

Claim 3.6. $g \in W_{\text{loc}}^{k,p}(\tilde{V}, \mathbb{R}^{m-r})$.

This statement is a direct consequence of the next

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^n$ be open, $p > n$ and $k \geq 1$. If $\Phi \in W_{\text{loc}}^{k,p}(\Omega, \mathbb{R}^n)$ is a diffeomorphism and $u \in W_{\text{loc}}^{k,p}(\Phi(\Omega))$, then $u \circ \Phi \in W_{\text{loc}}^{k,p}(\Omega)$.*

Proof. When $k = 1$ the result is obvious because diffeomorphisms preserve $W_{\text{loc}}^{1,p}$. Assume thus that $k \geq 2$. Since $p > n$, $\Phi \in C^{k-1}$ so Φ is a diffeomorphism of class C^{k-1} , but also $u \in C^{k-1} \subset C^1$ and hence the classical chain rule gives

$$(3.9) \quad D(u \circ \Phi) = ((Du) \circ \Phi) \cdot D\Phi.$$

Since $Du \in W_{\text{loc}}^{k-1,p}$ and Φ is a diffeomorphism of class C^{k-1} , we conclude that $(Du) \circ \Phi \in W_{\text{loc}}^{k-1,p}$. Now the fact that $D\Phi \in W_{\text{loc}}^{k-1,p}$ combined with (3.9) and Lemma 3.5 yield that the right hand side of (3.9) belongs to $W_{\text{loc}}^{k-1,p}$ so $D(u \circ \Phi) \in W_{\text{loc}}^{k-1,p}$ and hence $u \circ \Phi \in W_{\text{loc}}^{k,p}$. This completes the proof of Lemma 3.7 and hence that of Claim 3.6. \square

Now we can complete the proof of Claim 3.3. Recall that we need to prove that

$$(3.10) \quad \mathcal{H}^\ell((V \cap K_r) \cap f^{-1}(y)) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.$$

The diffeomorphism Y^{-1} is a change of variables that simplifies the structure of the mapping f because $f \circ Y^{-1}$ fixes the first r coordinates (see (3.8)) and hence it maps $(n-r)$ -dimensional slices orthogonal to \mathbb{R}^r to the corresponding $(m-r)$ -dimensional slices orthogonal to \mathbb{R}^r . Because of this observation it is more convenient to work with $f \circ Y^{-1}$ rather than with f . Translating (3.10) to the case of $f \circ Y^{-1}$ it suffices to show that

$$\mathcal{H}^\ell((\tilde{V} \cap Y(K_r)) \cap (f \circ Y^{-1})^{-1}(y)) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.$$

We used here a simple fact that the diffeomorphism Y preserves ℓ -null sets.

Observe also that

$$(3.11) \quad \text{rank } D(f \circ Y^{-1})(x) = r \quad \text{for } x \in \tilde{V} \cap Y(K_r).$$

For any $\tilde{x} \in \mathbb{R}^r$ and $A \subset \mathbb{R}^n$, we will denote by $A_{\tilde{x}}$ the $(n-r)$ -dimensional slice of A with the first r coordinates equal to \tilde{x} . That is, $A_{\tilde{x}} := \{z \in \mathbb{R}^{n-r} \mid (\tilde{x}, z) \in A\}$. Let $g_{\tilde{x}} : \tilde{V}_{\tilde{x}} \rightarrow \mathbb{R}^{m-r}$ be defined by $g_{\tilde{x}}(z) = g(\tilde{x}, z)$. With this notation

$$(f \circ Y^{-1})(\tilde{x}, z) = (\tilde{x}, g_{\tilde{x}}(z))$$

and hence for $y = (\tilde{x}, w) \in \mathbb{R}^m$

$$(\tilde{V} \cap Y(K_r)) \cap (f \circ Y^{-1})^{-1}(y) = g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r))_{\tilde{x}}.$$

More precisely the set on the left hand side is contained in an affine $(n - r)$ -dimensional subspace of \mathbb{R}^n orthogonal to \mathbb{R}^r at \tilde{x} while the set on the right hand side is contained in \mathbb{R}^{n-r} but the two sets are identified through a translation by the vector $(\tilde{x}, 0) \in \mathbb{R}^n$ which identifies \mathbb{R}^{n-r} with the affine subspace orthogonal to \mathbb{R}^r at \tilde{x} .

According to the Fubini theorem it suffices to show that for almost all $\tilde{x} \in \mathbb{R}^r$ the following is true: for almost all $w \in \mathbb{R}^{m-r}$

$$(3.12) \quad \mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r))_{\tilde{x}}) = 0.$$

As we will see this is a direct consequence of the induction hypothesis applied to the mapping $g_{\tilde{V}} : \tilde{x}_{\tilde{x}} \rightarrow \mathbb{R}^{n-r}$ defined in a set of dimension $n - r \leq n - 1$. We only need to check that $g_{\tilde{x}}$ satisfies the assumptions of the induction hypothesis.

It is easy to see that for each $x = (\tilde{x}, z) \in \tilde{V}$

$$D(f \circ Y^{-1})(x) = \begin{pmatrix} \text{id}_{r \times r} & 0 \\ * & D(g_{\tilde{x}})(z) \end{pmatrix}.$$

This and (3.11) imply that for each $\tilde{x} \in \mathbb{R}^r$, $Dg_{\tilde{x}} = 0$ on the slice $(\tilde{V} \cap Y(K_r))_{\tilde{x}}$. Hence the set $(\tilde{V} \cap Y(K_r))_{\tilde{x}}$ is contained in the critical set of $g_{\tilde{x}}$ so

$$(3.13) \quad \mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r))_{\tilde{x}}) \leq \mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap C_{g_{\tilde{x}}}).$$

It follows from the Fubini theorem applied to Sobolev spaces that for almost all $\tilde{x} \in \mathbb{R}^n$, $g_{\tilde{x}} \in W_{\text{loc}}^{k,p}(\tilde{V}_{\tilde{x}}, \mathbb{R}^{m-r})$ and hence the induction hypothesis is satisfied for such mappings

$$W_{\text{loc}}^{k,p} \ni g_{\tilde{x}} : \tilde{V}_{\tilde{x}} \subset \mathbb{R}^{n-r} \rightarrow \mathbb{R}^{m-r}.$$

Since

$$\ell = \max(n - m - k + 1, 0) = \max((n - r) - (m - r) - k + 1, 0),$$

for almost all $w \in \mathbb{R}^{m-n}$ the expression on the right hand side of (3.13) equals zero and (3.12) follows. This completes the proof of Claim 3.3 and hence that of the theorem. \square

REFERENCES

- [1] ABRAHAM, R., ROBBIN, J.: *Transversal Mappings and Flows*. W. A. Benjamin, Inc. (1967).
- [2] ALBERTI, G.: Generalized N -property and Sard theorem for Sobolev maps. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 23 (2012), no. 4, 477–491.
- [3] BOJARSKI, B., HAJLASZ, P., STRZELECKI, P.: Sard's theorem for mappings in Holder and Sobolev spaces. *Manuscripta Math.* 118 (2005), 383–397.
- [4] BOURGAIN, J., KOROBKOV, M.V., KRISTENSEN, J.: On the Morse-Sard property and level sets of Sobolev and BV functions. (English summary) *Rev. Mat. Iberoam.* 29 (2013), 1–23.
- [5] BOURGAIN, J., KOROBKOV, M.V., KRISTENSEN, J.: On the Morse-Sard property and level sets of $W^{n,1}$ Sobolev functions on \mathbb{R}^n . *J. Reine Angew. Math.* 700 (2015), 93–112.
- [6] DE PASCALE, L.: The Morse-Sard theorem in Sobolev spaces. *Indiana Univ. Math. J.* 50 (2001), no. 3, 1371–1386.
- [7] DUBOVITSKIĬ, A. YA: On the structure of level sets of differentiable mappings of an n -dimensional cube into a k -dimensional cube. *Izv. Akad. Nauk SSSR. Ser. Mat.* 21 (1957), 371–408.
- [8] EVANS, L., GARIEPY, R.: *Measure Theory and Fine Properties of Functions*. CRC Press, Inc. (1992).

- [9] FIGALLI, A.: A simple proof of the Morse-Sard theorem in Sobolev spaces. *Proc. Amer. Math. Soc.* 136 (2008), 3675–3681.
- [10] GILBARG, D., TRUDINGER, N. S.: *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
- [11] GLAESER, G.: Étude de quelques algèbres tayloriennes. *J. Analyse Math.* 6 (1958), 1–124.
- [12] GRINBERG, E. L.: On the smoothness hypothesis in Sard's theorem. *Amer. Math. Monthly* 92 (1985), 733–734.
- [13] HAJLASZ, P.: Whitney's example by way of Assouad's embedding. *Proc. Amer. Math. Soc.* 131 (2003), 3463–3467.
- [14] HAJLASZ, P., MALEKZADEH, S.: On conditions for unrectifiability of a metric space. *Anal. Geom. Metr. Spaces* 3 (2015), 1–14.
- [15] KAUFMAN, R. P.: A singular map of a cube onto a square. *J. Differential Geom.* 14 (1979), 593–594.
- [16] KNESER, M.: Abhängigkeit von Funktionen. *Mathematische Zeitschrift.* 54 (1951), 34–51.
- [17] KOROBKOV, M., KRISTENSEN, J.: On the Morse-Sard Theorem for the sharp case of Sobolev mappings. *Indiana Univ. Math. J.* 63 (2014), 1703–1724.
- [18] MALGRANGE, B.: *Ideals of differentiable functions*. Tata Institute of Fundamental Research (1966).
- [19] MALÝ, J., SWANSON, D., ZIEMER, W.: The co-area formula for Sobolev mappings. *Trans. Amer. Math. Soc.* 355 (2003), 477–492.
- [20] MARGALEF-ROIG, J., OUTERELO DOMINGUEZ, E.: *Differential Topology*. North-Holland Mathematics Studies, 173. North-Holland Publishing Co., Amsterdam, 1992.
- [21] METIVIER, M.: Valeurs critiques des applications différentiables. *Anais de Acad. Brasileira de Ciencias.* 36 (1964), 383–397.
- [22] MORSE, A. P.: The behavior of a function on its critical set. *Annals of Mathematics* 40 (1939), 62–70.
- [23] SARD, A.: The measure of the critical values of differentiable maps. *Bulletin of the American Mathematical Society* 48 (1942), 883–890.
- [24] STERNBERG, S.: *Lectures on differential geometry*. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.
- [25] THOM, R.: Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.* 28 (1954), 17–86.
- [26] VAN DER PUTTEN, R.: The Morse-Sard theorem in $W^{n,n}(\Omega)$: a simple proof. (English summary) *Bull. Sci. Math.* 136 (2012), no. 5, 477–483.
- [27] WHITNEY, H.: A function not constant on a connected set of critical points. *Duke Math J.* 1 (1935), no. 4, 514–517.
- [28] ZIEMER, W. P.: *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

P. HAJLASZ: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, 301 THACKERAY HALL, PITTSBURGH, PA 15260, USA, hajlasz@pitt.edu

S. ZIMMERMAN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, 301 THACKERAY HALL, PITTSBURGH, PA 15260, USA, srz5@pitt.edu